The Return of Newton-Cartan Spacetime

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based on work with:

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1409.1519 (PLB), 1409.1522 (PRD), 1502.00228 (JHEP)

Jelle Hartong 1504.0746 (JHEP)

Jelle Hartong, Marco Sanchioni 160x.yyyy (to appear)

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1311.4794 (PRD) & 1311.6471 (JHEP)
Outline

• Why Newton-Cartan (NC) ? (→ non-relativistic space-time)
  - holography,
  - field theory
  - gravity

• What is Newton-Cartan geometry ?
  - NC & its torsionful generalization (TNC) from gauging the Bargmann algebra

• Coupling of non-rel field theories to NC

• TNC in NR hydro (& non-relativistic fluid/gravity correspondence)

• What theory of gravity does one get when making TNC dynamical ?
  - connection to Horava-Lifshitz gravity & new perspectives

• Outlook
Motivation (Holography)

AdS/CFT has been very successful tool in studying strongly coupled (conformal) relativistic systems

-> holography beyond original AdS-setup ?

• How general is holographic paradigm ?
  (nature of quantum gravity, black hole physics, cosmology)

• Examples of potentially holographic descriptions based on non-AdS space-times:
  Lifshitz, Schrödinger, warped AdS3 (Kerr/CFT), flat space-time.

  - simplest example appears to be
  Lifshitz spacetimes
    \[ ds^2 = -\frac{dt^2}{r^{2z}} + \frac{1}{r^2} \left( dr^2 + d\vec{x}^2 \right) \]

  characterized by anisotropic (non-relativistic) scaling between time and space
    \[ t \to \lambda^z t, \quad \vec{x} \to \lambda \vec{x} \]

  [Kachru, Liu, Mulligan]

* introduced originally to study strongly coupled systems with critical exponent z
Motivation (Holography) cont’d

• for standard AdS setup: boundary geometry is Riemannian just like the bulk geometry

• not generic: in beyond-AdS holography boundary geometry typically non-Riemannian

-> need new approach: prime (simplest) example to gain traction = Lifshitz
   (lessons can subsequently be applied to other cases)

Result: torsional Newton-Cartan geometry is the boundary geometry found in large class of examples in EPD model

-> Lifshitz holography dual to field theories on TNC space-time

by making the resulting non-Riemannian geometry dynamical one gains access to other bulk theories of gravity (than those based on Riemannian gravity)
- apply holography (e.g. HL gravity)
- interesting in their own right
Different Holographic setups

bulk: Riemannian gravity (GR)

non-Riemannian gravity (e.g. HL gravity)

boundary: CFT

Riemannian geometry

AdS

non-AdS

NR-FT with scaling

NR-FT with scaling

non-Riemannian
Motivation (Field Theory)

• in relativistic FT: very useful to couple to background (Riemannian) geometry
  -> compute EM tensors, study anomalies, Ward identities, etc.

- background field methods for systems with non-relativistic (NR) symmetries require
  NC geometry (with torsion)

  -> there is full space-time diffeomorphism invariance when coupling to the
  right background fields

- Recent examples
  * Son’s approach to the effective field theory for the FQHE
    [Son, 2013], [Geracie, Son, Wu, Wu, 2014]
  * non-relativistic (NR) hydrodynamics
    [Jensen, 2014]
Motivation (Gravity)

- interesting to make NC geometry dynamical
  -> “new” theories of gravity

will see: dynamical Newton-Cartan (NC) = Horava-Lifshitz (HL) gravity

natural geometric framework with full diffeomorphism invariance & possibly non-trivial consequences for HL gravity

such theories of gravity interesting as
- other bulk theories of gravity in holographic setups
- effective theories (cond mat, cosmology)

- Galilean quantum gravity
  infinite c limit of $(\hbar, G_N, 1/c)$ cube
Newton-Cartan makes Galilean local

- NC geometry originally introduced by Cartan to geometrize Newtonian gravity
  Eisenhart, Trautman, Dautcourt, Kuenzle, Duval, Burdet, Perrin, Gibbons, Horvathy, Nicolai, Julia…..
  both Einstein’s and Newton’s theories of gravity admit geometrical formulations which are diffeomorphism invariant

- NC originally formulated in “metric” formulation
  more recently: vielbein formulation (shows underlying sym. principle better)
  Andringa, Bergshoeff, Panda, de Roo

Riemannian geometry: tangent space is Poincare invariant

Newton-Cartan geometry: tangent space is Bargmann (central ext. Gal.) invariant

- gives geometrical framework and extension to include torsion
  i.e. as geometry to which non-relativistic field theories can couple
  (boundary geometry in holographic setup is non-dynamical)

* will consider dynamical (torsional) Newton-Cartan later
Mini-review: From Poincare to GR by gauging

- make Poincare local (i.e. gauge the translations and rotations)

\[ A_\mu = P_a e^a_\mu + \frac{1}{2} J_{ab} \omega^{ab}_\mu \]
\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = P_a R^{a}_{\mu \nu}(P) + \frac{1}{2} J_{ab} R^{ab}_{\mu \nu}(J) \]
\[ \delta A_\mu = \partial_\mu \Lambda + [A_\mu, \Lambda], \quad \Lambda = \xi^\mu A_\mu + \frac{1}{2} J_{ab} \lambda^{ab} \]

\[ R^{a}_{\mu \nu}(P) = 0 \]
\[ \delta A_\mu \rightarrow \delta e^a_\mu = \mathcal{L}_\xi e^a_\mu + e^b_\mu \lambda^a_b \]

Spin connection expressed in terms of vielbein

- GR is a diff invariant theory whose tangent space invariance group is the Poincaré group
- Einstein equivalence principle -> local Lorentz invariance

**Postulate**

\[ R^{ab}_{\mu \nu}(J) = \text{Riemann curvature 2-form} \]
Relevant non-relativistic algebras

Galilean

\[ H, \, P_a, \, J_{ab}, \, G_a, \, N \]

(Galilean algebra is c to infinity limit of Poincare)

\[ [H, G_a] = P_a \quad [P_a, G_b] = 0 \]

Bargmann

\[ [P_a, G_b] = N\delta_{ab} \]

Lifshitz

\[ H, \, P_a, \, J_{ab}, \, D, \quad G_a, \, N, \, K(z = 2) \]

Schrödinger

\[ [D, H] = zH \quad [D, P_a] = P_a \]

\[ [D, N] = (2 - z)N \]

Schrödinger = Bargmann + dilatations (+ special conformal for z=2)
Gauging the Bargmann algebra

<table>
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<tr>
<th>symmetry</th>
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<tr>
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<tr>
<td>space translations</td>
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<td>$\lambda^{ab}(x^\nu)$</td>
<td>$R_{\mu\nu}^{ab}(J)$</td>
</tr>
<tr>
<td>central charge transf.</td>
<td>$N$</td>
<td>$m_\mu$</td>
<td>$\sigma(x^\nu)$</td>
<td>$R_{\mu\nu}(N)$</td>
</tr>
</tbody>
</table>

impose curvature constraints: (e.g.)

$$R_{\mu\nu}(H) = R_{\mu\nu}^a(P) = R_{\mu\nu}(N) = 0.$$  

independent fields:  $\tau_\mu, e^a_\mu, m_\mu$

= gauge fields of Hamiltonian, spatial translations and central charge

$$\delta \tau_\mu = \mathcal{L}_\xi \tau_\mu$$

$$\delta e^a_\mu = \mathcal{L}_\xi e^a_\mu + \lambda^a \tau_\mu + \lambda^a_b e^b_\mu$$

$$\delta m_\mu = \mathcal{L}_\xi m_\mu + \partial_\mu \sigma + \lambda_a e^a_\mu$$
Newton-Cartan geometry

NC geometry = no torsion

TTNC geometry = twistless torsion

TNC geometry

- in TTNC: torsion measured by $a_\mu = \mathcal{L}_\hat{\nu} \tau_\mu$

geometry on spatial slices is Riemannian

(gauge field of central extension of Galilean algebra (Bargmann))

(time)

(inverse) spatial metric:

$g^{\mu\nu} = e^\mu_a e^\nu_b \delta^{ab}$

notion of absolute time

preferred foliation in equal time slices
Adding torsion to NC

- inverse vielbeins

\[(\nu^\mu, e^\mu_a)\]

\[\nu^\mu \tau_\mu = -1, \quad \nu^\mu e^a_\mu = 0, \quad e^\mu_\mu \tau_\mu = 0, \quad e^\mu_a e^b_\mu = \delta^b_a\]

can build Galilean boost-invariants

\[\hat{\nu}^\mu = \nu^\mu - h^{\mu\nu} M_\nu,\]
\[\bar{h}_{\mu\nu} = h_{\mu\nu} - \tau_\mu M_\nu - \tau_\nu M_\mu,\]
\[\tilde{\Phi} = -\nu^\mu M_\mu + \frac{1}{2} h^{\mu\nu} M_\mu M_\nu,\]

-introduce Stueckelberg scalar \(\chi\)
(to ensure N-invariance):

\[M_\mu = m_\mu - \partial_\mu \chi.\]

affine connection of TNC (inert under G,J,N)

\[\Gamma^\rho_{\mu\nu} = -\hat{\nu}^\rho \partial_\mu \tau_\nu + \frac{1}{2} h^{\rho\sigma} \left( \partial_\mu \bar{h}_{\nu\sigma} + \partial_\nu \bar{h}_{\mu\sigma} - \partial_\sigma \bar{h}_{\mu\nu} \right)\]

with torsion \[\Gamma^\rho_{[\mu\nu]} = -\frac{1}{2} \hat{\nu}^\rho \left( \partial_\mu \tau_\nu - \partial_\nu \tau_\mu \right)\]

\[\nabla_\mu \tau_\nu = 0, \quad \nabla_\mu h^{\nu\rho} = 0,\]
analogue of metric compatibility
intermezzo: geodesics on NC space-time

- worldline action of non-rel particle of mass $m$ on NC background

$$ S = \int d\lambda L = \frac{m}{2} \int d\lambda \frac{\bar{h}_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}{\tau_{\rho} \dot{x}^\rho} $$

[Kuchar], [Bergshoeff et al]

- gives the geodesic equation with NC connection

$$ \frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\lambda} \frac{dx^\rho}{d\lambda} = 0, $$

* reduces to Newton’s law

$$ \frac{d^2 x^i}{dt^2} + \delta^{ij} \partial_j \Phi = 0, $$

provided we take

$$ M_t = \partial_t M + \Phi, $$
$$ M_i = \partial_i M, $$

for flat NC space-time: zero Newtonian potential

symmetries of flat NC = conformal Killing vectors (spanning Lifshitz) + extra
torsion in NC (recent activity)

- NC introduced in problem of FQH  
  Son (1306)

- TNC first observed as bdry geometry
  in $z=2$ Lifshitz holography
  & generalized to large class with general $z$  
  Christensen,Hartong,Rollier,NO (1311)
  Hartong,Kiritsis,NO (1409)

- TTNC introduced in FQH  
  Geracie,Son,Wu,Wu (1407)

- TNC from gauging Schrödinger algebra  
  Bergshoeff,Hartong,Rosseel (1409)

- TNC from gauging Bargmann (with torsion)  
  Hartong,NO (1504)

- coupling of non-relativistic field theories to TNC
  (independent of holography)  
  Jensen (1408)
  Hartong,Kiritsis,NO (1409)

- TNC related to warped geometry that couples to 2D WCFT  
  Hofmann,Rollier (1411)

- other approaches  
  Banerjee,Mitra,Mukherjee (1407),
  Brauner,Endlich,Monin,Penco (1407)
  Bekaert,Morand (1412),Geracie,Prabhu,Roberts (1503)

- recent activity using NC/TNC in CM
  (strongly-correlated electron system, FQH)  
  Gromov,Abanov, Moroz,Hoyos,Geracie,Son
  Wu,Wu,Geracie,Golkar,Roberts,....
  Jensen,Karch (1412)

- (T)NC from non-rel limits/super NC  
  Bergshoeff,Rosseel,Zojer (1505,1509,1512)
Coupling FTs to TNC

- action functional

\[ S = S[\phi^\mu, h^{\mu\nu}, \Phi]. \]

- EM tensor:

\[
\begin{array}{ll}
T^{\mu}_{\nu} & \text{mass current} \\
T^{\mu} & \text{momentum current}
\end{array}
\]

- energy current (density + flux)

\[
\delta S \sim \int d^{d+1}x e[\mathcal{E}^\mu \delta \tau_\mu + \mathcal{P}_\mu h^{\mu\nu} \delta v^\nu + T_{\mu\nu} h^{\mu\rho} h^{\nu\sigma} \delta h^{\rho\sigma} + T^\mu \delta m_\mu]
\]

- momentum current

- spatial stress

- mass density

* important to have torsion in order to describe the most general energy current!

- from the various local symmetries:

  - particle number conservation (if extra local U(1))

  - mass current = momentum current (local boosts)

  - symmetric spatial stress (local rotations)
Diffeomorphism and scale Ward identities

- **diffeos** -> on-shell WI

\[
\nabla_\nu T^\nu_\mu + \text{torsion terms} + \rho \nabla_\mu \tilde{\Phi} = 0
\]

* conserved currents \( \partial_\nu (eK^\mu T^\nu_\mu) = 0 \).

for \( K \) a **TNC Killing vector**: extra force term

- if theory has **scale invariance**:
  can use **TNC analogue of dilatation connection**

\[
ze + \text{Tr} T_{\text{spatial}} + 2(z - 1)\rho \Phi = 0
\]

z-deformed trace WI
Schrödinger model & Lifshitz model

- simplest toy model for coupling non-rel. scale-inv theory to TNC (z=2)

\[ S = \int d^{d+1}x \left( -i\phi^* \hat{\nabla}^\mu \partial_\mu \phi + i\phi \hat{\nabla}^\mu \partial_\mu \phi^* - h^{\mu\nu} \partial_\mu \phi \partial_\nu \phi^* - 2\Phi \phi \phi^* - V_0(\phi\phi^*)^{\frac{d+2}{d}} \right) \]

-> gives Schr. equation

* can also consider deformations preserving local scale inv

- other possibility: do not couple to \( \Phi \) -> e.g. z=2 Lifshitz model

\[ S = \int d^{d+1}x \left[ \frac{1}{2} (\hat{\nabla}^\mu \partial_\mu \phi)^2 - \frac{\lambda}{2} (h^{\mu\nu} \nabla_\mu \partial_\nu \phi)^2 \right] \]
Flat NC space-time & conserved currents

- Riemannian:
  symmetries of flat space (i.e. Minkowski) = Poincare (or conformal)
  -> relativistic field theories on Minkowski are Poincare invariant

notion of flat NC:
- use global inertial coordinates \((t, x^i)\)
  \[ \Gamma^\rho_{\mu\nu} = 0 \rightarrow M_\mu = \partial_\mu M. \]

- symmetries of flat NC = conformal Killing vectors (spanning Lifshitz) + extra

- flat NC space should include \(M=\text{const.}\)
  * turns out that one can allow for more general choices that are equivalent
to \(M=\text{const}\) by local syms of the theory
  -> defines the notion of orbit of \(M\) (depends on matter couplings, see later)

* interplay between conserved currents and space-time isometries is different
  compared to relativistic case: same mechanism seen in Lifshitz holography!

  -> in spirit: analogous to fact that relativistic field theories that are also conformal
  need special couplings
TNC in NR hydro (& fluid/gravity correspondence)

- TNC of growing interest in cond-mat (str-el, mes-hall) literature

  developments in Lifshitz holography can drive development of tools to study dynamics and hydrodynamics of non-rel. systems

  Lifshitz hydro: [Hoyos,Kim,Oz]
  Galilean: [Jensen]

  (in parallel to progress in the last many years in relativistic fluids and superfluids inspired from the fluid/gravity correspondence in AdS)

TNC right ingredients to start constructing effective TNC theories and their coupling to matter (e.g. QH-effect, cosmology,…)

- organizing principle for derivative expansion of stress tensor/mass current (transport coefficients)
- non-relativistic fluid/gravity correspondence:
  consider boosted Lifshitz black branes & perturb

[Kiritsis,Matsuo], [Hartong,NO,Sanchioni](in progress)
ingredient: TNC by dimensional reduction

- can get TNC structures and WIs from dim reduction

\[
ds^2 = \gamma_{AB} dx^A dx^B = 2\tau_\mu dx^\mu (du - m_\nu dx^\nu) + h_{\mu\nu} dx^\mu dx^\nu
\]

\[
\gamma^{uu} = 2\tilde{\Phi}, \quad \gamma^{u\mu} = -\hat{\psi}^\mu, \quad \gamma^{\mu\nu} = h^{\mu\nu}
\]

\[-\] relation between higher dim. EM tensor \( t^{AB} \) and lower dim. EM tensor & mass current

\[
t^{\mu u} = 2\tilde{\Phi} T^\mu - \hat{\psi}^\sigma T^\mu_\sigma,
\]
\[
t^{\mu\nu} = -\hat{\psi}^{\mu\nu} + h^{\mu\rho} T^\nu_\rho.
\]

diffeo Ward identity

\[
\nabla_A t^A_\nu = \tilde{\nabla}_\mu T^{\mu\nu} + 2\tilde{\Gamma}^\rho_{[\mu\rho]} T^{\mu\nu} - 2\tilde{\Gamma}^{\mu}_{[\nu\rho]} T^\rho_\mu + \tau_\mu T^{\mu\nu} \partial_\nu, = 0
\]
Perfect non-relativistic fluids from null reduction

• start with relativistic perfect fluid

\[ t_{AB} = (E + P) U_A U_B + P \gamma_{AB} \]
\[ U_A U^A = -1. \]

\[ U_u^2 = \frac{\rho}{E + P}, \]
\[ h^{\mu\nu} U_\nu = U_u (\hat{v}^\mu - u^\mu) \]
\[ \hat{v}^\mu U_\mu = \frac{1}{2} U_u (\hat{h}_{\mu\nu} u^\mu u^\nu + 2 \Phi + U_u^{-2}) \]

gives EM tensor & mass current in TNC covariant form
(including contribution from Newton potential)

\[ T^{\mu\nu} = \left( E + P + \rho \Phi + \frac{1}{2} \rho \hat{h}_{\lambda\kappa} u^\lambda u^\kappa \right) u^\mu \tau_\nu + P \delta^{\mu}_{\nu} + \rho u^\mu \hat{h}_{\nu\rho} u^\rho \]
\[ T^\mu = -\rho u^\mu. \]

scale invariance:

\[ z \left( E - \rho \Phi \right) - dP + \frac{z - 2}{2} \rho \left( \hat{h}_{\mu\nu} u^\mu u^\nu - 2 \Phi \right) = 0, \]

[Hartong,NO,Sanchioni](in progress)
Dynamical Newton-Cartan geometry

so far: (T)NC geometry was non-dynamial:
- what happens when we allow it to fluctuate?

• what is the theory of gravity that incorporates local Galilean symmetry?
  (Einstein equivalence principle, but applied to Galilean instead of Lorentz)

recently shown that:

  [Hartong,NO]1504

  • dynamical NC geometry = projectable HL gravity

  • dynamical TTNC geometry = non-projectable HL gravity

* Horava-Lifshitz gravity was originally introduced as non-Lorentz invariant and renormalizable UV completion of gravity
- phenomenologically viable?
- interesting theoretically as alternate bulk gravity theories
  relevant to i) holography for strongly coupled non-relativistic systems
    ii) alternate theories in cosmology
NC/TTNC gravity

TNC geometry is a natural geometrical framework underlying HL gravity

- NC quantities combine into: \[ g_{\mu\nu} = -\tau_\mu \tau_\nu + \hat{h}_{\mu\nu} \]
- ADM parametrization of metric used in HL gravity: Horava (0812,0901)
  \[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + \gamma_{ij} (dx^i + N^i dt) (dx^j + N^j dt) \]
relation:
  \[ \tau_\mu \sim \text{lapse}, \quad \hat{h}_{\mu\nu} \sim \text{spatial metric}, \quad m_\mu \sim \text{shift + Newtonian potential} \]
some features:

- khronon field of BPS appears naturally Blas,Pujolas,Sibiryakov(2010)
  \[ \tau_\mu = \psi \partial_\mu \tau \]
NC (no torsion):
  \[ N = N(t) \] projectable HL gravity
TTNC:
  \[ N = N(t,x) \] non-projectable HL gravity
- U(1) extension of HMT emerges naturally as Bargmann U(1)
- new perspective (via chi field) on nature of U(1) symmetry Horava,Melby-Thompson(2010)
Effective actions reproduce HL

- covariant building blocks:
  - extrinsic curvature: \( \hat{h}_{\nu\rho} \nabla_\mu \hat{\nu}^\rho = -K_{\mu\nu} \)
  - spatial curvature \( R_{\mu\nu\sigma} \)
  - covariant derivative, torsion vector \( a_\mu \)
  - inverse spatial metric \( h^{\mu\nu} \)
  - tangent space invariant integration measure \( e = \det(\tau_\mu, e^a_\nu) \)

-> construct all terms that are relevant or marginal (up to dilatation weight \( d+z \))
- in 2+1 dimensions for \( 1 < z \leq 2 \)

\[
S = \int d^3x e \left[ C \left( h^{\mu\rho} h^{\nu\sigma} K_{\mu\nu} K_{\rho\sigma} - \lambda (h^{\mu\nu} K_{\mu\nu})^2 \right) - V \right]
\]

potential:
\[
- V = 2\Lambda + c_1 h^{\mu\nu} a_\mu a_\nu + c_2 \mathcal{R} + \delta_{z,2} \left[ c_{10} (h^{\mu\nu} a_\mu a_\nu)^2 + c_{11} h^{\mu\rho} a_\mu a_\rho \nabla_\nu (h^{\nu\sigma} a_\sigma) \\
+ c_{12} \nabla_\nu (h^{\mu\rho} a_\rho) \nabla_\mu (h^{\nu\sigma} a_\sigma) + c_{13} \mathcal{R}^2 + c_{14} \mathcal{R} \nabla_\mu (h^{\mu\nu} a_\nu) + c_{15} \mathcal{R} h^{\mu\nu} a_\mu a_\nu \right]
\]

action also obtained by other method: [Afshar, Bergshoeff, Mehra, Parekh, Rollier]1512
Perspectives for HL gravity

HL originally formulated as dynamical theory with Riemannian metric & invariance under foliation-preserving diffeomorphisms (FPdiff)

-> reformulation: dynamical (torsional) Newton-Cartan geometry & invariance under full diffeomorphism group

new perspectives on HL:

- different vacuum (flat NC space-time): reexamine issues with HL gravity
- IR effective theory for non-relativistic field theories
- insights into non-relativistic quantum gravity corner of $[\hbar, G_N, 1/c]$ cube?

• black holes in HL/NC gravity

• relevance for cosmology? couple covariant (T)NC gravity EOMs to covariant non-relativistic perfect fluid stress tensor

• examine TNC gravity (general torsion)
  * relation with vector khronon of [Janiszewski,Karch]
Coupling of HL/NC gravity to FTs & models of NCG

have tools/building blocks to covariantly couple HL/NC gravity to non-relativistic field theories:
- massless/massive spin \((0),1/2,1,3/2\) for no-torsion using non-relativistic contraction:  \[\text{[Bergshoeff,Rosseel,Zojer]1512}\]
- spin \(1/2\) for generic NC using non-rel limit:  \[\text{[Fuini,Karch,Uhlemann]1510}\]
- GED (Galilean electrodynamics) coupled to TNC  \[\text{[Festuccia,Hansen,Hartong,NO](to appear)}\]

relation to other approaches (involving non-commutative geometry):

both gravity & matter with FPdiff: can be achieved using spectral action principle  \[\text{[Chamsedinne,Connes]}\]
implemented for fermionic matter:  \[\text{[Lopes,Mamiya,Pinzul]1508}\]
both sectors governed by same generalized Dirac operator

\[S = \text{Tr} f \left( \frac{D^2}{\Lambda^2} \right) + \langle J \psi, D \psi \rangle \equiv S_{\text{geom}} + S_{\text{matt}}\]
Outlook

- NC supergravity, NC in string theory

- revisit HL gravity using TNC language/connections with NR String Theory

- Carollian (ultra-relativistic) gravity (gauging of Caroll group) relation to flat space holography

- non-relativistic field theories from contracting backgrounds/null reduction

- applications to non-rel. hydrodynamics: fluid/gravity: black branes with zero/non-zero particle no. ? Galilean perfect fluids

- employ similar techniques to Schrödinger, warped AdS, flat space holography

- adding charge (Maxwell in the bulk) and Galilean electrodynamics adding other exponents (hyperscaling, matter scaling)

- applications to CMT (e.g. QH-effect) & cosmology
The end